Dissipative system with asymmetric interaction and Hopf bifurcation

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A dissipative system with asymmetric interaction, as well as the optimal velocity model, generally shows a Hopf bifurcation concerned with the transition from homogeneous motion to the formation of nontrivial patterns. We reveal that the origin of Hopf bifurcation in macroscopic phenomena is strongly related to asymmetric interaction in a microscopic many-body system, using the continuum system derived from the original discrete system.

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I. INTRODUCTION

Over the last few decades, physicists have shown growing interest in complex systems as many-body systems of simple components. The society formed by people, the collective biomotion formed by organisms, the granular media formed by particles, and the traffic flow formed by vehicles—these are examples of complex systems. One of the most interesting subjects is the pattern formation caused by cooperative effects in many-body systems [1]. We focus on the formation of a traffic jam of a vehicular flow and phenomena related to the cluster formation of a particle flow [2].

As a mathematical model for describing such phenomena, we investigate the optimal velocity model (the OV model), which is first introduced as a model for a traffic flow in 1994 [3,4]. The model well reproduces actual data of highway traffic [5]. From the physical point of view, the model is a nonequilibrium dissipative system describing a one-dimensional chain of interacting particles formulated by nonlinear equations. The interaction between particles in the OV model is asymmetric, meaning that a particle interacts with the particle in front in the direction of motion, not with the particle behind. This interaction breaks the action-reaction principle and the momentum conservation law is not preserved. The several interesting dynamical properties are originated in the asymmetry of interactions [6]. The model has two kinds of solutions: a homogeneous flow solution and a moving-cluster solution. If a control parameter exceeds a certain critical value, a homogeneous flow solution becomes unstable and a stable moving-cluster solution appears. As a model for traffic flow, a jam cluster emerges beyond the critical vehicle density.

The change in the stability from a homogeneous flow to a cluster flow is caused by the collective effect in many-body systems. The phenomenon is so called a dynamical phase transition. Another property is a bifurcation in dynamical systems. This property is an important characteristic in nonequilibrium dissipative systems of OV-type models. In many numerical simulations, we observe the profile of a jam flow solution as a kind of limit cycle in the phase space of headway and velocity [4]. The appearance of the profile indicates that the transition in the OV model is a Hopf bifurcation [7]. However, there are few analytical studies on this issue [8–11]. In this paper, we analytically show that the transition between the solutions is a Hopf bifurcation, and that it originates from asymmetric interactions. For this purpose, we derive the continuum system from the original OV model and investigate the property of the transition, instead of the original discrete system. This paper is organized as follows. We first review the OV model briefly in Sec. II. Next, we derive the continuum system and carry out a linear stability analysis of the system and investigate the property of a Hopf bifurcation in Sec. III. In Sec. IV, we investigate the general asymmetric interaction of dissipative systems of OV-type models, and we prove that a Hopf bifurcation originates from asymmetric interactions. Section V is devoted to the summary and discussion

II. BRIEF REVIEW OF OV MODEL

We briefly review the basic features of the OV model [3,4]. The model describes a one-dimensional particle-following system, where N particles move on a circuit with the length L. We express the equation of motion for the nth particle $(n=1,2,\dots,N)$ as

$$\ddot{x}_n = a\{V(\Delta x_n) - \dot{x}_n\}, \tag{1}$$

where x_n denotes the position of the nth particle and Δx_n is the headway defined by $\Delta x_n = x_{n+1} - x_n$. The overdot represents the time derivative. The parameter a is a sensitivity constant $(a>0)$. The function $V(\Delta x_n)$ is the so-called optimal velocity function, which monotonically increases and has an upper bound for $\Delta x_n \to \infty$.

Equation (1) has a homogeneous flow solution expressed as

$$x_n(t) = bn + V(b)t + \text{const}, \tag{2}$$

where b is an average distance expressed as $b=L/N$. By linear stability analysis, the solution is unstable under the condition that there exists a mode $\theta = n\pi/N$ satisfying the inequality [3]

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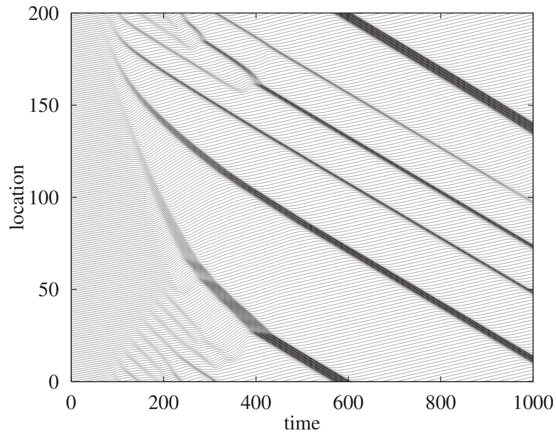

FIG. 1. Space-time plot of cluster formation of the OV model with OV function as $V(\Delta x)=\tanh(\Delta x-2)+\tanh 2$ in $N=100$ on the periodic boundary condition on a circuit. The vertical axis is the location on the circuit. The horizontal axis is the time evolution.

$$\cos^2\frac{\theta}{2} \geq \frac{a}{2V'(b)}, \tag{3}$$

where $V'(b)$ is the derivative of V at b. For a Fourier mode satisfying Eq. (3), its eigenvalue for the coefficient of time is real and positive, which blows up an amplitude of oscillation and makes the trivial solution unstable.

The equality of Eq. (3) with $\theta\to 0$ gives the critical condition $a=2V'(b)$ for the stability of the homogeneous flow solution. The condition predicts the critical vehicle density for given a. The change in the stability is a phase transition in many-body systems. In the case $a<2V'(b)$, the solution of the homogeneous flow is unstable and decays. Instead, the moving-cluster solution emerges and becomes stable as in Fig. 1. After relaxation, the cluster flow solution is stable. All clusters are moving backward with the same velocity opposed to the direction of the particle motion.

We recognize the profile of cluster flow solution by the trajectory of particles in the phase space of headway and velocity $(\Delta x_n, \dot{x}_n)$ in Fig. 2. In the cluster flow solution, all particles are moving along the specific closed curve, which is a kind of limit cycle [4,6]. Figure 2 indicates that the transition can be understood as a bifurcation in dynamical systems. The profile like a limit cycle naturally reminds us a Hopf bifurcation in dynamical systems. In the following section, we deal with the transition for the change in the stability from the viewpoint of dynamical systems.

III. CONTINUUM SYSTEM OF OV MODEL AND HOPF BIFURCATION

We investigate the property of the bifurcation in the OV model. From the result of numerical simulations, we expect that the transition is a Hopf bifurcation. The transition is caused by a long-wavelength mode of eigenvalue for the linearized equation of motion. For the analysis of this mode, the continuum system of the original model is more convenient to treat.

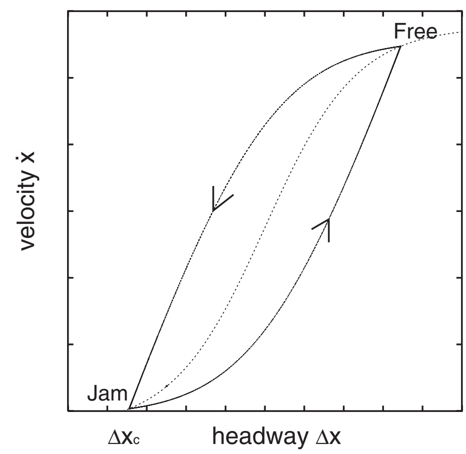

FIG. 2. The profile of a cluster flow solution. "Free" and "jam" denote smoothly moving regions and jam clusters, respectively. All vehicles move along the closed loop in the direction of arrow. A dotted curve represents the OV function.

A. Continuum system

We derive a continuum system of the OV model as follows. First, we transform Eq. (1) to the equation for the headway Δx_n by subtracting x_{n+1} from x_n and rewrite the equation using a deviation from the average headway distance b as a dynamical variable, $r_n=\Delta x_n-b$,

$$\ddot{r}_n = a\{V(r_{n+1}+b) - V(r_n+b) - \dot{r}_n\}. \tag{4}$$

In this formula, the homogeneous flow solution (2) is translated to $r_n(t)=0$.

Next, we use the shift operator $(\exp\frac{\partial}{\partial n})f(n)=f(n+1)$ by treating the index of a particle number n as the continuous variable. We replace $r_n(t)$ with $r(x,t)$, where x is the continuous variable defined by $x=bn$ by taking the continuum limit as $N\to\infty$, $L\to\infty$, at fixed $b=L/N$.

Then, Eq. (4) is rewritten as

$$\frac{\partial^2 r(x,t)}{\partial t^2} = a\left\{\left(\exp b\frac{\partial}{\partial x} - 1\right)V[r(x,t)+b] - \frac{\partial r(x,t)}{\partial t}\right\}, \tag{5}$$

We have derived the continuum system expressed by the partial differential equation (PDE) [8], for the original OV model formulated by the set of ordinary differential equations (ODEs) for many particles (1).

We investigate the linear stability of the trivial solution $r(x,t)=0$ for Eq. (5), which corresponds to the homogeneous flow solution of the original system. The linearized equation for the small deviation from the trivial solution is

$$\frac{\partial^2 r(x,t)}{\partial t^2} = a\left\{V'(b)[r(x+b,t) - r(x,t)] - \frac{\partial r(x,t)}{\partial t}\right\}, \tag{6}$$

where $V'(b)$ is the derivative of V at b. The solution of Eq. (6) is obtained by Fourier transformation. The wave number k takes continuous values in $(-\infty,\infty)$. The solution for the mode $\theta=kb$ is written as

$$r(x,t) = e^{ikx}e^{zt}, \tag{7}$$

where $z=\sigma-i\omega$ is the eigenvalue for each θ. The real part σ and the imaginary part ω take real values. From Eq. (6), the eigenvalue z satisfies

$$z^2 + az - aV'(b)(e^{i\theta} - 1) = 0 \tag{8}$$

for each mode θ. Thus, σ and ω satisfy the following relations:

$$\sigma^2 - \omega^2 = aV'(b)(\cos\theta - 1) - a\sigma, \tag{9}$$

$$-2\sigma\omega = aV'(b)\sin\theta + a\omega. \tag{10}$$

From Eqs. (9) and (10), we obtain the solutions $\sigma_\pm(\theta)$ and $\omega_\pm(\theta)$ for θ as

$$\sigma_\pm(\theta) = -\frac{1}{2}a \pm \frac{1}{2\sqrt{2}}\left\{a^2 - 8aV'(b)\sin^2\frac{\theta}{2} + \left\{a^4 + 16a^2V'(b)[4V'(b) - a]\sin^2\frac{\theta}{2}\right\}^{1/2}\right\}^{1/2}, \tag{11}$$

$$\omega_\pm(\theta) = \mp aV'(b)\sin\theta / \frac{1}{\sqrt{2}}\left\{a^2 - 8aV'(b)\sin^2\frac{\theta}{2} + \left\{a^4 + 16a^2V'(b)[4V'(b) - a]\sin^2\frac{\theta}{2}\right\}^{1/2}\right\}^{1/2}. \tag{12}$$

We can obtain the stability condition for the trivial solution of the continuum system by calculating $\sigma_+(\theta)\geq 0$ for (11) [8], which mode θ induces the instability and makes the amplitude blow up with time evolution. The obtained condition provides just the same result (3) as the stability condition for the homogeneous flow solution of the original discrete system.

Here, we remark that the continuum system or the macroscopic model corresponding to the original discrete system (the many-particle system) of the OV model should be carefully derived in order to conserve the important properties—the asymmetry of interaction in the discrete particles. This property cannot be introduced by a naive continuum limit, such as $V(\Delta x_n)\to V(\rho(x))$, where $\rho(x)$ is a local density. If one does such a crude approximation, he needs some modifications to preserve the dynamical effect of the asymmetric interaction in the original particle system [12,13]. However, our derivation of the continuum system is straightforward and the most faithful to the original OV model. So, the stability condition is reproduced correctly, and other investigations of the complex dynamical properties can be correctly carried out.

B. Hopf Bifurcation in OV model

Now, we investigate the property of the bifurcation in the OV model. We consider a as the control parameter for the bifurcation. The critical sensitivity for a given b is denoted by $a_c=2V'(b)$. For the proof of existence of Hopf bifurcation in the OV model, we analyze each eigenvalue $z(\theta)$ for a mode θ in the continuum system.

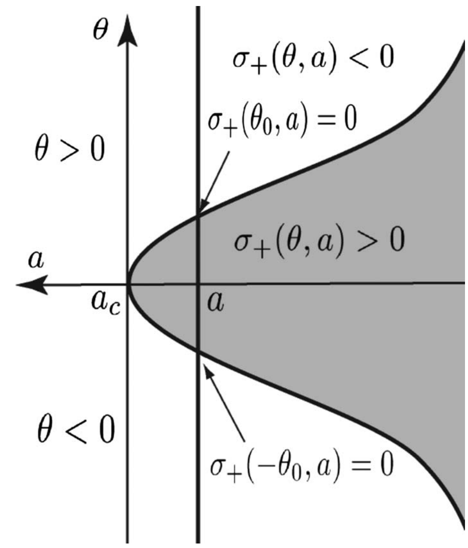

FIG. 3. The sketch of solutions $\sigma_+(\theta)$, the real part of the eigenvalue $z(\theta)$. Solutions $\sigma_+(\theta)$ for a given a denoted by $\sigma_+(\theta,a)$ in the figure are on the vertical line of a. The curve $f(\theta,a)=0$ is drawn. The solutions are positive in the shaded area. $z(\pm\theta_0)$ for the mode θ_0 are complex conjugate pure imaginary.

We first remark that $z(\pm\theta)=\sigma(\theta)\mp i\omega(\theta)$ are complex conjugate eigenvalues. We can restrict $\theta\geq 0$, because the mode different in only the sign plays the same role in the following investigation. And, it is enough to analyze the real part of the eigenvalue $z_+=\sigma_+(\theta)-i\omega_+(\theta)$ for $\theta>0$ to study the instability and the transition, because only σ_+ has the possibility to be positive for some θ, and the mode for such θ contributes the instability and a bifurcation. For the purpose of our analysis, we prepare $f(\theta,a)$ defined as

$$f(\theta,a) = -\frac{[a - a_c\cos^2(\theta/2)]}{a_c} \simeq \frac{a_c - a}{a_c} - \frac{\theta^2}{4}, \tag{13}$$

where the approximate equality holds for small θ. The function $f(\theta,a)$ is continuous and monotonically decreases in terms of θ and a. We can see easily that the sign of $\sigma_+(\theta)$, which is the real part of the eigenvalue $z(\theta)$, is equivalent to that of $f(\theta,a)$ (see the Appendix).

In the case of $a>a_c$, the sign of $\sigma_+(\theta)$ is negative and nonvanishing for any mode θ, which is easily seen by $f(\theta,a)$. Then, the trivial solution is stable and no Hopf bifurcation.

In the case of $a<a_c$, by the continuity and the monotonicity of $f(\theta,a)$, there exists a mode $\theta=\theta_0$ such that $\sigma_+(\theta)$ is negative for $\theta>\theta_0$ and positive for $\theta<\theta_0$. Namely, $\sigma_+(\theta_0)=0$. The solution of the imaginary part $\omega_+(\theta_0)$ satisfies Eqs. (9) and (10) with $\sigma=0$. These relations are reduced to the equation of ω^2, which has the positive solution $\omega_+^2=a(a_c-a)$. Then, the eigenvalue of the mode θ_0 is pure imaginary, and the eigenvalues $z(\pm\theta_0)$ are complex conjugate pure imaginary (Fig. 3). This means a Hopf bifurcation in the mode θ_0 for a given $a(<a_c)$. For a mode $\theta<\theta_0$, $\sigma_+(\theta)$ is positive, which blows up an amplitude of oscillation and makes the trivial solution unstable.

For $a(<a_c)\rightarrow a_c$, θ_0 decrease to zero and the region of unstable modes $\theta<\theta_0$ reduces to zero and the mode for Hopf bifurcation vanishes. Thus, $a=a_c$ is the "critical" point for existing a Hopf bifurcation. In this meaning, a_c can be identified as the "Hopf bifurcation point" for a given b. This situation is a little bit different from a usual type of Hopf bifurcation in dynamical systems, but it is curious and interesting as a Hopf bifurcation in many-body systems.

As the critical point of the original discrete system is not different from that of the continuum system, there exists the Hopf bifurcation point a_c for a given b in the original discrete system. The direct investigation for a Hopf bifurcation using the original many-particle system was previously studied by Gasser *et al.* [9]. We have provided a simple proof using the continuum system. The proof can be applicable to a dissipative system with more general asymmetric interaction in the OV-type model to have a Hopf bifurcation.

IV. ASYMMETRIC INTERACTION AND HOPF BIFURCATION

We have seen that the OV model has a Hopf bifurcation in the previous section. We investigate the origin of Hopf bifurcation in the OV model. The model has the interaction $V(\Delta x_n)$ describing that a particle interacts only with the particle in front in the direction of motion, meaning that the interaction is asymmetric. We clarify that a Hopf bifurcation generally occurs in the dissipative system with asymmetric interactions in the type of the OV model.

We generalize the asymmetric interaction in the OV model by introducing a term $W(\Delta x_{n-1})$, which represents an interaction with the particle behind as [14]

$$\ddot{x}_n = a\{V(\Delta x_n) - W(\Delta x_{n-1}) - \dot{x}_n\}. \tag{14}$$

If we take $W(\Delta x)=V(\Delta x)$, Eq. (14) represents a dissipative system of many particles with usual (namely, symmetric) interactions, where the momentum conservation is preserved. The system describes a one-dimensional chain of oscillators with nonlinear interactions and viscosity term, for example. We can compare a usual (symmetric) system with an asymmetric system like the OV model for many aspects in dynamical properties [15].

We derive the corresponding continuum system to Eq. (14) in the same way as that in the OV model,

$$\frac{\partial^2 r}{\partial t^2} = a\left\{\left(\exp b\frac{\partial}{\partial x} - 1\right)V(r+b) - \left[1-\exp\left(-b\frac{\partial}{\partial x}\right)\right]W(r+b) - \frac{\partial r}{\partial t}\right\}. \tag{15}$$

Then, the real part σ and the imaginary part ω of the eigenvalue $z=\sigma-i\omega$ for each Fourier mode solution (7) of the linearized equation of PDE (15) are satisfied in the following relations:

$$\sigma^2 - \omega^2 = a[V'(b)+W'(b)](\cos\theta - 1) - a\sigma, \tag{16}$$

$$-2\sigma\omega = a(V'(b)-W'(b))\sin\theta + a\omega. \tag{17}$$

We obtain the solutions, the real part $\sigma_\pm(\theta)$, and the imaginary part $\omega_\pm(\theta)$ for θ as

$$\sigma_\pm(\theta) = -\frac{1}{2}a \pm \frac{1}{2\sqrt{2}}\left\{a^2 - 8a[V'(b)+W'(b)]\sin^2\frac{\theta}{2} + \left\{a^4 + 16a^2[V'(b)+W'(b)]\{4[V'(b)+W'(b)]-a\}\times\sin^2\frac{\theta}{2} - 16^2a^2V'(b)W'(b)\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\right\}^{1/2}\right\}^{1/2}, \tag{18}$$

$$\omega_\pm(\theta) = \mp a[V'(b)-W'(b)]\sin\theta \Bigg/ \frac{1}{\sqrt{2}}\left\{a^2 - 8a[V'(b) + W'(b)]\sin^2\frac{\theta}{2} + \left\{a^4 + 16a^2[V'(b)+W'(b)]\times\{4[V'(b)+W'(b)]-a\}\sin^2\frac{\theta}{2} - 16^2a^2V'(b)W'(b)\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\right\}^{1/2}\right\}^{1/2}. \tag{19}$$

The trivial solution is unstable for the mode θ satisfying the condition [14]

$$\cos^2\frac{\theta}{2} \geq \frac{a[V'(b)+W'(b)]}{2[V'(b)-W'(b)]^2}. \tag{20}$$

We denote $\tilde{a}^c$ as the critical sensitivity for a given b expressed as

$$\tilde{a}_c = \frac{2[V'(b)-W'(b)]^2}{V'(b)+W'(b)}. \tag{21}$$

Now, we investigate the property of a Hopf bifurcation in the dissipative system with a generalized asymmetric interaction. First, we consider the case of a symmetric interaction, $W(b)=V(b)$. Equation (17) or Eq. (19) obviously shows that $\omega(\theta)=0$ is always satisfied for any θ. Then, the eigenvalue $z(\theta)$ has no imaginary part. Therefore, the system has no Hopf bifurcation points.

Although, in the case of asymmetric interactions, $W(b)\neq V(b)$, the situation is different from that in the symmetric case. For proof of Hopf bifurcation, we investigate the sign of σ_+ in Eq. (18) in the same way as in Sec. III B. For this purpose we use again the function $f(\theta,a)$ by replacing a_c with $\tilde{a}^c$ (see the Appendix),

$$f(\theta,a) = -\frac{[a-\tilde{a}_c\cos^2(\theta/2)]}{\tilde{a}_c} \simeq \frac{\tilde{a}_c - a}{\tilde{a}_c} - \frac{\theta^2}{4}. \tag{22}$$

The same process as that in the previous proof is followed. There exists a mode θ_0 such that the sign of $\sigma_+(\theta)$ changes in the vicinity of θ_0 for a given $a<\tilde{a}_c$. The eigenvalue $z(\theta_0)$ is pure imaginary, which means a Hopf bifurcation. This situation is possible for $W(b)\neq V(b)$ by seeing Eqs. (16) and (17) with $\sigma=0$. In this case the equations can be reduced to the equation of ω^2, which has the positive solution, as the same as the case of the OV model. For $a(<\tilde{a}_c)\rightarrow\tilde{a}_c$, the mode for Hopf bifurcation vanishes. Thus, in this meaning $\tilde{a}_c$ is the Hopf bifurcation point for a given b. And this phenomenon is

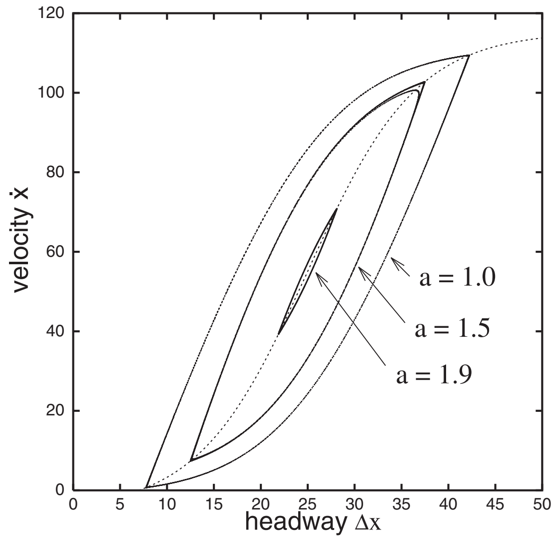

FIG. 4. Profile of jam flow solution for various values of the sensitivity $a=1.0, 1.5, 1.9$ with $b=b^{*}=2$ in headway-velocity space in the OV model. The OV function is chosen as $V(\Delta x)=\tanh(\Delta x-2)+\tanh 2$. In this case, the Hopf bifurcation point is $a_c^{*}=2$.

originated from the asymmetry of $W(\Delta x_n)$ and $V(\Delta x_n)$.

Consequently, a Hopf bifurcation can occur in dissipative systems with asymmetric interactions. The OV model and a generalized OV-type model are very simple systems for such examples.

V. SUMMARY AND DISCUSSION

Let us summarize the studies in this paper. We investigate the property of transition from a homogeneous flow solution to a moving-cluster solution in the OV model. For this purpose, we derive the continuum system from the original discrete system of particles. Our formula of the continuum system is well expressed preserving the dynamical properties of the original particle system.

We prove analytically that the transition of the OV model is a Hopf bifurcation. Moreover, we investigate dissipative systems with asymmetric interactions generalized from the OV model. The property of transition in such systems is also a Hopf bifurcation, which originates from asymmetric interactions between particles.

We overview our next studies as follows. In dynamical systems, a Hopf bifurcation usually leads to a limit cycle. Actually, in the OV model the moving-cluster solution is identified as a kind of limit cycles in the headway-velocity space, as shown in Fig. 2. The size of the limit cycle depends on the parameter a as shown in Fig. 4 [5]. The limit cycle shrinks as $a \to a_c^{*}$ for b^{*}, where $b=b^{*}$ is the inflection point of the OV function, $V''(b^{*})=0$. We perform a nonlinear analysis to investigate the dynamical property of the limit cycle, which determines the details of transition in the Hopf bifurcation. Actually, whether its type is supercritical or subcritical depends on b.

For this purpose, we introduce a dynamical system under the assumption of a traveling wave, $r(x,t) \equiv r(x-ct)$, in a continuum system (5), where c is the velocity of a moving cluster corresponding to the limit cycle depending on a. Then, the dynamical system expressed in a PDE (5) is rewritten to an ODE for a traveling wave. In order to define the derived ODE system, we should determine the velocity $c(a)$ of a moving cluster, which can be evaluated by the property of limit cycles in the original discrete system of particles in the OV model. First, we show that the Hopf bifurcation point of the ODE system is just the Hopf bifurcation point of the PDE system studied in this paper. And, we can determine the velocity of the cluster $c(a_c)$ at the Hopf bifurcation point. Then, we carry out the analysis by expansion in terms of $o(a-a_c)$ for a given b in order to construct the normal form to investigate the limit cycle.

There are many nonequilibrium dissipative phenomena, for example, Belousov-Zhabotinsky reactions [16] and Rayleigh-Bénard convections [17]. They are usually described by macroscopic models expressed in PDE and show Hopf bifurcations, as well as the continuum system for the OV model. We expect the possibility that some class of a Hopf bifurcation observed in a macroscopic phenomenon indicates the existence of asymmetric interaction in an underlying microscopic nonequilibrium dissipative system.

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APPENDIX: EQUIVALENCE OF THE SIGN BETWEEN $\sigma_{+}(\theta)$ AND $f(\theta,a)$

In Eq. (11), by defining A, B as $A \equiv a^2 - 8aV'(b)\sin^2\frac{\theta}{2}$, $B \equiv 8aV'(a)\sin\frac{\theta}{2}\cos\frac{\theta}{2}$, we rewrite $\sigma_{+}(\theta) \gtreqless 0$ as

$$\sigma_{+}(\theta) = -\frac{1}{2}a + \frac{1}{2}\sqrt{\frac{A+\sqrt{A^2+B^2}}{2}} \gtreqless 0. \tag{A1}$$

Inequality (A1) is equivalent to the following inequality:

$$\sqrt{A^2+B^2} \gtreqless a^2 + 8a\sin^2\frac{\theta}{2}. \tag{A2}$$

As both sides of inequality (A2) are positive, the square of each side holds the same inequality, which is equivalent to

$$\left(8aV'(a)\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2 \gtreqless 32a^3V'(b)\sin^2\frac{\theta}{2}. \tag{A3}$$

Then, we obtain that $\sigma_{+}(\theta) \gtreqless 0$ is equivalent to

$$\cos^2\frac{\theta}{2} \gtreqless \frac{a}{2V'(b)}. \tag{A4}$$

We define $f(\theta,a)=\cos^2\theta/2 - a/2V'(b)$, with $a_c=2V'(b)$. Thus, $\sigma_{+}(\theta) \gtreqless 0$ is equivalent to $f(\theta,a) \gtreqless 0$.

For the proof in the case of a general asymmetric interaction, we redefine $f(\theta,a)$ as follows. In Eq. (18), by defining A, B as $A\equiv a^2-8a[V'(b)+W'(b)]\sin^2\frac{\theta}{2}$, $B\equiv 8a[V'(a)-W'(b)]\sin\frac{\theta}{2}\cos\frac{\theta}{2}$, we can write down an inequality in the same form as in Eq. (A1). Then, in the same way we obtain that $\sigma_+(\theta)\gtreqless 0$ is equivalent to

$$\cos^2\frac{\theta}{2}\gtreqless\frac{a[V'(b)+W'(b)]}{2[V'(b)-W'(b)]^2}. \tag{A5}$$

Then, we redefine $f(\theta,a)$ by using $\tilde{a}$ expressed as Eq. (21).

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